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THE RELATIONS OF DEFORMATION THEORY IN THE QUADRATIC APPROXIMATION AND THE PROBLEMS OF CONSTRUCTING IMPROVED VERSIONS OF THE GEOMETRICALLY NON-LINEAR THEORY OF LAMINATED STRUCTURES†

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Two versions of the approximate relations of the deformation theory of continuous media, known as the complete version (see, for example [1]) and the incomplete version (see, for example [2, 3]) of the quadratic approximation of the non-linear theory are analysed.It is shown that the relations of the complete version, which define the elongation deformation, and the relations of the incomplete version, which define the shear deformation, are incorrect, since, when solving specific problems, they lead to the occurrence of false bifurcation points. For small elongation deformation and medium shear deformation a non-contradictory version of the kinematic relations is constructed in the quadratic approximation, representing a combination of the relations of the complete and incomplete versions. The simplest examples of its application, connected with the reduction of the two-dimensional non-linear problem of the deformation of a strip in the form of a rod to homogeneous equations and their subsequent use to detect possible forms of loss of stability for characteristic forms of loading them, are considered. Essentially new results are obtained connected with the investigation of forms of loss of stability of a rod under uniform transverse compression and pure shear. In the first case the behaviour of the load turns out to be important: if it remains normal to the deformation axis of the rod, bifurcation is only possible with respect to the shear form, if it retains its direction, and then, in addition to bifurcation with respect to the shear form, a bending form of loss of stability is possible, which is identical in form with the Euler form, for which there are no shears. In the second case, i.e. when there is a load which causes pure shear of the rod, to investigate its bifurcation values, it is necessary to describe the shear deformation by non-linear kinematic relations in the complete quadratic version, whereas when there are no subcritical shear stresses one can use the simplified relations. An example of the investigation of the forms of loss of stability of a circular ring when acted upon by a uniform external pressure having zero variability in the circumferential direction is also considered.

1. RELATIONS OF DEFORMATION THEORY IN THE QUADRATIC APPROXIMATION

If the space of a body in the initial (undeformed) state is referred to rectangular Cartesian coordinates x, y, z, and we denote the components of the displacements by u, υ, w, then for arbitrary displacements for elongation deformation E_x, E_y, E_z and shear deformation $\sin\gamma_{xy}$, $\sin\gamma_{xz}$, $\sin\gamma_{yz}$ we have the formulae

$$E_x = (1+2\varepsilon_{xx})^{1/2}-1, \ldots; \quad \sin\gamma_{xy} = (1+2\varepsilon_{xx})^{-1/2}(1+2\varepsilon_{yy})^{-1/2}\varepsilon_{xy}, \ldots \tag{1.1}$$

by which, using the six components of the deformation

$$\varepsilon_{xx} = u_{,x}+\frac{1}{2}(u_{,x}^2+\upsilon_{,x}^2+w_{,x}^2), \ldots; \quad \varepsilon_{xy} = u_{,y}+\upsilon_{,x}+u_{,x}u_{,y}+\upsilon_{,x}\upsilon_{,y}+w_{,x}w_{,y}, \ldots \tag{1.2}$$

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the arbitrary deformed state of the body is described. An extremely clear derivation of formulae (1.1) and (1.2) and a comprehensive analysis of them has been given by Novozhilov [1]; these formulae can be seen at the present time in practically any scientific and educational literature on the non-linear theory of elasticity.

If the elongation deformation is small, i.e. $E_x \approx \varepsilon \ll 1$, from the first group of relations of (1.1) we obtain the following relations of undoubted rigor when the degree of accuracy $2 + E_x \approx 2$ is satisfied

$$E_x \approx \varepsilon_{xx} = u_{,x} + \frac{1}{2}(u_{,x}^2 + v_{,x}^2 + w_{,x}^2), \ldots \tag{1.3}$$

and, when an accuracy of $(1 + 2E_x)^{-1/2} \approx 1$ from the second group of relations of (1.1) we obtain

$$\sin\gamma_{xy} \approx \varepsilon_{xy} = u_{,y} + v_{,x} + u_{,x}u_{,y} + v_{,x}v_{,y} + w_{,x}w_{,y}, \ldots \tag{1.4}$$

or for small shear angles γ_{xy}, γ_{xz}, γ_{yz}

$$\gamma_{xy} \approx \varepsilon_{xy} + u_{,y} + v_{,x} + u_{,x}u_{,y} + v_{,x}v_{,y} + w_{,x}w_{,y}, \ldots \tag{1.5}$$

These relations were derived in [1] together with formulae (1.3) and are widely used as kinematic relations in the quadratic approximation.

Donnell [2] also derived another incomplete quadratic version of the kinematic relations, when $E_x, \ldots, \sin\gamma_{xy}, \ldots$ are calculated, apart from squares and pair wise products of the derivatives of the displacements, from the formulae

$$E_x = u_{,x} + \frac{1}{2}(v_{,x}^2 + w_{,x}^2), \ldots \tag{1.6}$$

$$\sin\gamma_{xy} = u_{,y} + v_{,x} - u_{,x}v_{,x} - u_{,y}v_{,y} + w_{,x}w_{,y}, \ldots \tag{1.7}$$

Shklyarchuk [3] also considered a simpler version of relations (1.7)

$$\gamma_{xy} = u_{,y} + v_{,x} + w_{,x}w_{,y}, \ldots \tag{1.8}$$

The need to estimate the quality of the above three versions of the kinematic relations in the quadratic approximation arose in connection with the appearance of false bifurcation points when solving specific problems, which were formulated in [4, 5], starting from relations of the form (1.3). One of these estimates can be obtained by considering uniaxial extension-compression and pure shear. A brief analysis of this kind was given earlier in [6].

2. UNIAXIAL EXTENSION-COMPRESSION

It was shown in [4] that for uniaxial extension-compression of a rod with a force P, applied at its ends, formulae (1.6) are preferable to formulae (1.3); moreover, in this case formulae (1.6) can be used for any elastic deformations. This can be shown as follows.

The variation of the work done by the forces applied to an elementary parallelepiped, when had dimensions dx, dy and dz before deformation and was deformed by a load along the x axis, according to Novozhilov will be equal to

$$\delta dA = \sigma_{xx}^*\delta\varepsilon_{xx}dxdydz; \quad \sigma_{xx}^* = \frac{S_x^*}{S_x}\frac{\sigma_{xx}}{(1+2\varepsilon_{xx})^{1/2}}$$
$$S_x^* = [(1+2\varepsilon_{yy})(1+2\varepsilon_{zz}) - \varepsilon_{yz}^2]^{1/2}dydz, \quad S_x = dydz \tag{2.1}$$

Then

$$\delta dA = (\sigma_{xx}S_x^*)\delta[(1+2\varepsilon_{xx})^{1/2} - 1]dx = dP_x\delta(E_xdx) \tag{2.2}$$

Here dP_x is the normal force applied to the face $dydz$ of the element.

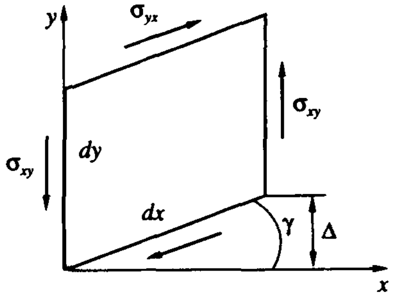

Fig. 1

The last expression could have been written immediately as the principle for possible displacements for a deformed element of length $(1 + E_x)dx = dx + u'dx$ $(u' = du/dx)$; then

$$E_x = u' \tag{2.3}$$

Expression (1.6) just leads to formula (2.3), which in the case considered, in view of the equalities $\upsilon_{,x} = w_{,x} \approx 0$, also follows when the first expression of (1.2) is substituted into the first equality of (1.1).

As the same time, approximation (1.3) gives

$$E_x \approx \varepsilon_{xx} = u' + u'^2/2 \tag{2.4}$$

which in the problem of the compression of a rod by a force P leads to the false bifurcation value $P = EF$ (E is the modulus of elasticity for stretching and F is the cross-section area) – and "absurd" loss of stability of the rod [4].

3. PURE SHEAR

For pure shear in the xy plane the variation of the work according to Novozhilov will be equal to

$$\delta dA = \sigma_{xy}^* \delta\varepsilon_{xy} dxdydz; \quad \sigma_{xy}^* = \frac{S_x^*}{S_x}\frac{\sigma_{xy}}{1+E_y}, \quad 1 + E_y = (1 + 2\varepsilon_{yy})^{1/2} \tag{3.1}$$

In Fig. 1 we show one of the possible positions of an element with respect to the coordinate axes; other positions differ only by a rotation as a rigid body. For the version represented $u = u(x)$, $\upsilon = \upsilon(x)$. Then the displacements should be such that $E_x = E_y = 0$; they are easily obtained:

$$\upsilon = x\sin\gamma, \quad u = x(\cos\gamma - 1) \tag{3.2}$$

It can be shown that then $E_x = E_y = 0$, $S_x^* = S_x$. Moreover, for pure shear $\sigma_{xx} = \sigma_{yy} = 0$, $\sigma_{xy}^* = \sigma_{xy}$. As a result we have

$$\delta dA = \sigma_{xy}\delta\varepsilon_{xy} dxdydz \tag{3.3}$$

If we write δdA according to the virtual displacement principle, then for pure shear by virtue of the fact that $\sigma_{xx} = 0$ we have $\tau_{xy} = \sigma_{xy}$ and

$$\delta dA = \sigma_{xy}\delta(dx\sin\gamma)dydz \tag{3.4}$$

Comparing expressions (3.3) and (3.4) it can be seen that they are identical if we take formulae (1.4) for ε_{xy}; in this case $\varepsilon_{xy} = \sin\gamma$ for any γ.

If we take relations (1.7), we have

$$\sin\gamma_{xy} = \sin\gamma - (\cos\gamma - 1)\sin\gamma \approx \sin\gamma + \sin^3\gamma/2 \tag{3.5}$$

In this case the same situation is possible as for compression, when the use of approximation (3.3) leads to the occurrence of a false bifurcation point in the solution [4].

Hence, the mixed version of the kinematic relations in the quadratic approximation gives the best approximation for the elementary states, when the elongation deformation is calculated from formulae (1.6) (according to Donnell) while the shear deformation is calculated from formulae (1.4) (according to Novozhilov).

Finally, we must draw attention to the fact that, as can be seen from Fig. 1, the energetically matched generalized displacement for τ_{xy} will be $\sin\gamma = \sin\gamma_{xy}$, rather than $\gamma = \gamma_{xy}$, i.e. the measure of the shear deformation is $\sin\gamma$ and not γ. True, this observation only relates to the form of the representation of Hooke's law for shear and to the problem of processing the corresponding experimental data for considerable shear deformation. Hence, for a linearly elastic material for large shear deformation, Hooke's law must be represented in the form (G is the shear modulus) $\tau = G\sin\gamma$, rather than in the form $\tau = G\gamma$, as is assumed everywhere.

Taking this observation into account, relations (1.8) must be written in the form

$$\sin\gamma_{xy} = u_{,y} + v_{,x} + w_{,x}w_{,y} + \ldots \tag{3.6}$$

Then the version of the combination of relations (1.6) with (3.6) will also not be contradictory.

For small shears $\sin\gamma_{xy} \approx \gamma_{xy}$, and relations (1.6) in combination with (1.8) will then also not be contradictory.

4. REDUCTION OF THE TWO-DIMENSIONAL KINEMATIC RELATIONS TO ONE-DIMENSIONAL RELATIONS FOR A RECTANGULAR STRIP (A ROD) BASED ON AN IMPROVED TIMOSHENKO MODEL

In order to estimate the quality of the kinematic relations derived it is useful to consider the simplest examples of their application, related to reducing the two-dimensional non-linear problem of the deformation of a strip to one-dimensional equations and their subsequent use to investigate possible forms of loss of stability for characteristic forms of loading.

We will assume that constant forces, having a value per unit length of p_x, p_y and p_{xy}, are applied to the edges of a rectangular strip, having dimensions of a and $2h$, as shown in Fig. 2. We will use the following approximations for the displacements U and V

$$U = u(x) + y\gamma(x), \quad V = v(x) + y\varphi(x) \tag{4.1}$$

which are well-known in the theory of single-layer and multilayer shells (the improved Timoshenko model taking transverse compression into account). Here $u(x)$ and $v(x)$ are the displacements of points on the x axis of the strip.

In the two-dimensional problem considered we have for the elongation deformation

$$E_x = e_{11} + e_{12}^2/2, \quad E_y = e_{22} + e_{21}^2/2; \quad e_{11} = u_{,x}, \quad e_{12} = v_{,x}, \quad e_{13} = w_{,x}, \ldots \tag{4.2}$$

and the following formula holds for the shear deformation

$$\sin\gamma_{12} = e_{12}(1 + e_{22}) + e_{21}(1 + e_{11}) \tag{4.3}$$

when using the complete kinetic relations (1.4), and the formula

$$\gamma_{12} = e_{12} + e_{21} \tag{4.4}$$

when using the incomplete relations (1.6).

Within the framework of representations (4.1) we have for the quantities e_{ij} occurring in formulae (4.2)–(4.4)

$$e_{11} = u' + y\gamma', \quad e_{22} = \varphi, \quad e_{12} = v' + y\varphi', \quad e_{21} = \gamma \tag{4.5}$$

When using them for a strip with the thin-walled parameter $2h/a = \varepsilon \ll 1$, by formulae (4.2) we arrive at the reduced kinematic relations

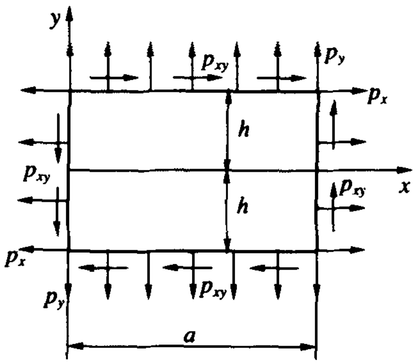

Fig. 2

$$E_x = E_x^0 + y\chi + y^2\varphi'^2/2 \approx E_x^0 + y\chi, \quad E_y = \varphi + \gamma^2/2 \tag{4.6}$$

where

$$E_x^0 = u' + v'^2/2, \quad \chi = \gamma' + v'\varphi' \tag{4.7}$$

Assuming that, in view of the relation $2h/a = \varepsilon \ll 1$, the approximation $\gamma_{12} = \gamma_{12}(x)$ holds, to determine the shear deformation we will use the reduced formula

$$\sin\gamma_{12} \approx (1+\varphi)v' + (1+u')\gamma \tag{4.8}$$

which is obtained by substituting expressions (4.5) into relation (4.3) in the complete quadratic approximation, and the formula in the linear approximation

$$\gamma_{12} = v' + \gamma \tag{4.9}$$

which is obtained using relation (4.4) in the incomplete quadratic approximation.

5. THE ONE-DIMENSIONAL IMPROVED EQUILIBRIUM EQUATIONS FOR A STRIP AND ITS FORM OF LOSS OF STABILITY FOR DIFFERENT FORMS OF LOADING

Corresponding to the approximations (4.6) and the approximations $\gamma_{12} = \gamma_{12}(x)$, we will introduce into consideration the following forces and moments per unit length

$$N_x = \int_{-h}^{h}\sigma_x dy, \quad M = \int_{-h}^{h}\sigma_x y dy, \quad N_y = \int_{-h}^{h}\sigma_y dy, \quad N_{xy} = \int_{-h}^{h}\tau_{xy} dy \tag{5.1}$$

reduced to the axial line x, and we will represent the variation of the potential energy of deformation of the strip in the form

$$\delta U = \int_0^a [N_x\delta E_x^0 + M\delta\chi + N_y\delta E_y + N_{xy}\delta(\sin\gamma_{12})]dx \tag{5.2}$$

and when using approximations (4.1) for the variation of the work of external forces, applied to the edges of the strip (Fig. 2), we arrive at the expression

$$\delta A = (2hp_x\delta u + 2hp_{xy}\delta v)\Big|_{x=0}^{x=a} + \int_0^a (2hp_y\delta\varphi + 2hp_{xy}\delta\gamma)dx \tag{5.3}$$

The equilibrium equations and the static boundary conditions at the edges of the strip $x = 0$ and $x = a$, when using expressions (5.2) and (5.3) can be obtained starting from the variational equation

$$\delta U - \delta A = 0 \tag{5.4}$$

Henceforth it will be more convenient to represent them for different degrees of approximation.

The equations and bifurcations of their solutions, corresponding to the use of approximation (4.9) and the Timoshenko model ignoring transverse compression. This model of deformation is the simplest of the existing improved models, in accordance with which it is necessary to put

$$\sigma_y = N_y \equiv 0 \tag{5.5}$$

In addition to relations (5.5) we also put

$$\varphi \equiv 0 \tag{5.6}$$

which, within the framework of assumption (5.5) is not fundamental.

By virtue of assumption (5.5) we can establish the following physical relations

$$N_x = B_1 E_x^0 = B_1(u' + v'^2/2), \quad B_1 = 2hE_1 \tag{5.7}$$

$$M = D_1 \chi = D_1 \gamma', \quad D_1 = 2h^3 E_1/3 \tag{5.8}$$

$$N_{xy} = B_{12}\gamma_{12} = B_{12}(v' + \gamma), \quad B_{12} = 2hG_{12} \tag{5.9}$$

where E_1 is the modulus of elasticity in the x direction and G_{12} is the shear modulus in the xy plane.

With these assumptions, starting from relations (4.7), (4.9) and (5.2)–(5.4) we can obtain the equilibrium equations

$$N_x' = 0, \; Q_y' = (N_{xy} + N_x v')' = 0, \quad M' - N_{xy} + 2hp_{xy} = 0 \tag{5.10}$$

and the boundary conditions at the edges $x = 0$ and $x = a$

$$N_x = 2hp_x \text{ for } \delta u \neq 0, \quad N_{xy} + N_x v' = 2hp_{xy} \text{ for } \delta v \neq 0, \quad M = 0 \text{ for } \delta\gamma \neq 0 \tag{5.11}$$

Since the component equations are only non-linear with respect to one unknown function v (the bending), it is convenient to use them to consider solely the case of pure axial compression of a strip by a force per unit length $p_x = -p$.

In this case of loading, the boundary conditions N_x can be written in the form

$$N_x = -2hp \text{ when } x = 0 \text{ and } x = a$$

by virtue of which the integral of the first equation of (5.8) will be the function

$$N_x = B_1(u' + v'^2/2) = -2hp \tag{5.12}$$

For the second equation of (5.10) to boundary conditions can be formulated in the form

$$v = 0 \text{ when } x = 0 \text{ and } x = a \tag{5.13}$$

if both end sections are fixed in the y direction, or in the form

$$v = 0 \text{ when } x = 0, \quad N_{xy} - 2hpv' = 0 \text{ when } x = a \tag{5.14}$$

if the right end section can be freely displaced in the y direction.

When using (5.9) the second equation of (5.10) can be written in the form

$$N_{xy} = B_{12}(v' + \gamma) = 2hpv' + C_1 \tag{5.15}$$

where C_1 is a constant of integration. Hence we obtain

$$\gamma = (C_1 + 2hpv')/B_{12} - v' \tag{5.16}$$

and, substituting expression (5.15) into the last equation of (5.10) and using relation (5.8), we arrive at the equation

$$D_1\gamma'' - 2hpv' = C_1$$

and its first integral. Introducing expression (5.16) into this first integral we obtain the resolvent of the problem (C_2 is a constant of integration)

$$D_1\left(1 - \frac{2hp}{B_{12}}\right)v'' + 2hpv = -C_1x - C_2 \tag{5.17}$$

The boundary conditions are formulated in the form of (5.3) or (5.4).

The general solution of Eq. (5.17), in addition to C_1 and C_2, contains two constants of integration. The following boundary conditions can be used to determine them

$$\gamma'(x = 0) = 0, \quad \gamma'(x = a) = 0$$

which occur in the case of the hinged support of the end sections.

The problems formulated above and the equations describing them, which are set up with the degree of accuracy indicated above, are completely equivalent to the problems and their solutions analysed in detail by Vasil'yev [7]. Without dwelling on their investigation, we note that they have two bifurcation values of the load $2hp$. One of these is given by the formula ($2hp_E$ is the well-known Euler critical load)

$$2hp_*^u = \frac{2hp_E}{1 + 2hp_E/B_{12}}$$

This corresponds to a bending form of the loss of stability of the rod and is obtained taking into account the transverse shear. (As pointed out by Timoshenko [8, p. 147], the effect of a shearing force on the critical force was first indicated by Engesser in 1891.) The following critical load corresponds to the second bifurcation point

$$p_*^c = G_{12}$$

on reaching which a purely shear form of loss of stability occurs.

Note that a discussion of questions related to this form of loss of stability of rods under axial compression can be found in the book [7] and also in other publications ([9], etc.). Nevertheless, it is important in principle to emphasise that to investigate a purely shear form of loss of stability under conditions of pure unilateral compression of a strip (a rod) it is sufficient to confine oneself to using the simplest kinematic Timoshenko model, based on: (1) a consideration of the shear deformation in the xy plane within the framework of the corresponding kinematic relation only in the linear approximation and (2) neglect of the normal stress and deformations in a direction orthogonal to the direction of the compression.

Note that for other forms of loading, when $N_x \equiv 0$, problems formulated starting from relations (5.7)–(5.11) are linear, and their solutions have no bifurcation points.

The equations and bifurcations of their solutions corresponding to the use of approximation (4.9), taking the stress σ_y and the deformation E_y into account. In order to simplify the calculations and to carry out solely a qualitative analysis of the problems considered above without losing their compactness, we will use the relations of Hooke's law in the form

$$\sigma_x = E_1E_x, \quad \sigma_y = E_2E_y, \quad \tau_{xy} = G_{12}\gamma_{12} \tag{5.18}$$

which corresponding to a hypothetical material with zero values of Poisson's ratios. Then, by formulae (5.18), (5.1), (4.9), (4.6) and (4.7) we arrive at the elasticity relations

$$\begin{aligned} &N_x = B_1E_x^0 = B_1(u' + v'^2/2), \quad N_y = B_2(\varphi + \gamma^2/2), \quad B_2 = 2hE_2 \\ &N_{xy} = B_{12}(v' + \gamma), \quad M = D_1\chi = D_1(\gamma' + v'\varphi') \end{aligned} \tag{5.19}$$

In the case considered, when using approximation (4.9) and relations (4.6) and (4.7) established for E_y, starting from relations (5.2) and (5.4) we obtain the equilibrium equations

$$\begin{aligned} &N_x' = B_1(u' + v'^2/2)' = 0, \quad Q_y' = (N_x v' + N_{xy} + M\varphi')' = 0 \\ &M' - N_y\gamma - N_{xy} + 2hp_{xy} = 0, \quad (Mv')' - N_y + 2hp_y = 0 \end{aligned} \tag{5.20}$$

and the boundary conditions when $x = 0$ and $x = a$

$$\begin{aligned} &N_x = 2hp_x \text{ when } \delta u \neq 0, \quad N_x v' + N_{xy} + M\varphi' = 2hp_{xy} \text{ when } \delta v \neq 0 \\ &M = 0 \text{ when } \delta\gamma \neq 0, \quad Mv' = 0 \text{ when } \delta\varphi \neq 0 \end{aligned} \tag{5.21}$$

On the basis of the above equations we will consider two forms of loading of the strip.

Suppose the strip is under conditions of pure shear due to the action of forces $p_{xy} = \tau$ per unit length, shown in Fig. 2.

To carry out a qualitative analysis we will introduce the standard assumption that the strip, before the loss of stability, is under tension, but is not deformed. With this assumption, in the initial unperturbed state we will have for the internal forces and moments introduced into consideration

$$N_x^0 = 0, \quad N_y^0 = 0, \quad N_{xy}^0 = 2h\tau, \quad M^0 = 0 \tag{5.22}$$

Linearizing Eqs (5.20) in the neighbourhood of the solution (5.22), and retaining the same notation used above for the increments of the variables, we obtain linearized equations of the perturbed state

$$\begin{aligned} &N_x' = B_1 u'' = 0, \quad B_2\varphi = 0, \quad N_{xy}' = B_{12}(v' + \gamma)' = 0 \\ &M' - N_{xy} = D_1\gamma'' - B_{12}(v' + \gamma) = 0 \end{aligned} \tag{5.23}$$

for which the boundary conditions, linearized in the neighbourhood of the solution (5.22), are uniform:

$$u' = 0 \text{ for } \delta u \neq 0, \quad N_{xy} = B_{12}(v' + \gamma) = 0 \text{ for } \delta v \neq 0, \quad \gamma' = 0 \text{ for } \delta\gamma \neq 0 \tag{5.24}$$

Equations (5.23) with boundary conditions (5.24) have only a trivial solution. Consequently, with the degree of accuracy assumed in describing the shear deformation by linear relation (4.9), the solutions of Eqs (5.20) for pure shear do not enable us to determine purely shear forms of loss of stability.

Suppose the strip is under conditions of unilateral compression by a force $p_x = -q$ in the transverse direction.

In the case considered, we will have the following solution for the initial unperturbed state instead of (5.22)

$$N_x^0 = 0, \quad N_y^0 = -2hq, \quad N_{xy}^0 = 0, \quad M^0 = 0 \tag{5.25}$$

and the equations of the perturbed state, linearized in its neighbourhood, will take the form

$$N_x' = 0, \quad B_2\varphi = 0, \quad N_{xy}' = 0, \quad M' - N_{xy} + 2hq\gamma = 0 \tag{5.26}$$

As in the previous case, the first and second equations of (5.26) have only a trivial solution.

Using the third equality of (5.26) we can eliminate N_{xy} from the last equation of (5.26). We obtain

$$M'' + 2hq\gamma' = 0$$

Hence, taking Hooke's law (5.8) into account, we obtain the equation

$$M'' + k^2 M = 0, \quad k^2 = 2hq/D_1 \tag{5.27}$$

the general solution of which has the form

$$M = C_1 \sin kx + C_2 \cos kx \tag{5.28}$$

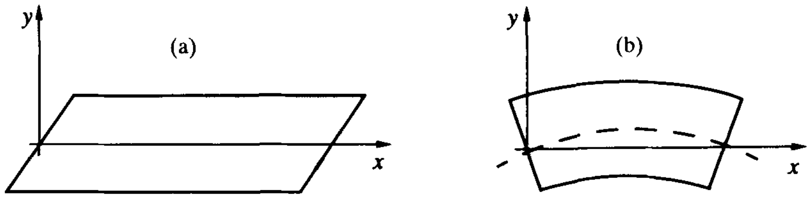

Fig. 3

For homogeneous boundary conditions

$$M(x=0) = 0, \quad M(x=a) = 0 \tag{5.29}$$

we arrive at the system of equations

$$C_2 = 0, \quad C_1 \sin kx = 0$$

which has two solutions.

1. $C_1 = C_2 = 0$, to which responds $M \equiv 0$ and, as can be seen from the last equation of (5.26),

$$N_{xy} - 2hq\gamma = 0 \tag{5.30}$$

Hence, taking Hooke's law (5.9) into account, we have

$$B_{12}\upsilon' + (B_{12} - 2hq)\gamma = 0 \tag{5.31}$$

This equation, when

$$q_*^{(1)} = \frac{B_{12}}{2h} = G_{12} \tag{5.32}$$

allows of the solution $\upsilon' = 0$, $\gamma \neq 0$, which corresponds to the occurrence of mixed shear forms of equilibrium while preserving rectilinearity of the x axis. The corresponding form of loss of stability is shown in Fig. 3(a).

2. $C_2 = 0$, $C_1 \neq 0$, $\sin ka = 0$, i.e. $ka = n\pi$ ($n = 1, 2, \ldots$). Hence, when $n = 1$ we obtain the critical load

$$q_*^{(2)} = \frac{\pi^2 D_1}{2ha^2} \tag{5.33}$$

To this there corresponds

$$M = C_1 \sin\frac{\pi x}{a}, \quad C_1 \neq 0 \tag{5.34}$$

and, by virtue of Hooke's law (5.8).

$$\gamma = \frac{C_1}{D_1}\cos\frac{\pi x}{a} + C_3 \tag{5.35}$$

In order to eliminate rotation of the strip as a rigid body, it is sufficient to prevent its middle section from rotating, i.e. to put $\gamma(x = a/2) = 0$. Hence it follows that $C_3 = 0$. We obtain the corresponding form of bending υ from the third equation of (5.26), taking Hooke's law (5.9) into account, which leads to the equation

$$(\upsilon' + \gamma)' = 0$$

Finally, taking expression (5.35) into account we obtain

$$\upsilon = -\frac{C_1 a}{D_1 \pi}\sin\frac{\pi x}{a} + C_3 + C_4 x \tag{5.36}$$

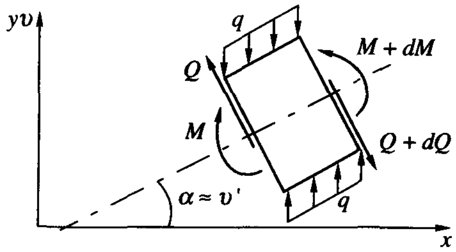

Fig. 4

If the ends of the strip axis are secured from vertical displacements, i.e. $\upsilon(x = 0) = \upsilon(x = a) = 0$, then $C_3 = C_4 = 0$, and then the shift $\gamma_{12} = \upsilon' + \gamma \equiv 0$. Hence, with this form of loss of stability the cross-section remains perpendicular to the deformed axis, and the strip behaves as a Bernoulli–Euler beam. The corresponding form of loss of stability is shown in Fig. 3(b).

It should be noted that a similar result can be obtained if we consider the strip as a Bernoulli–Euler beam, compressed in the transverse direction by loads q, which do not change its direction for bending perturbations. An element dx of such a beam in the perturbed state is shown in Fig. 4. The equilibrium equations of the element and the Hooke's law relation take the form

$$Q' = 0, \quad M' - Q + 2hq\alpha = 0, \quad M = D_1\chi = D_1\alpha' = D_1\upsilon'' \tag{5.37}$$

Equation (5.27) immediately follows from these, from which, for the same boundary conditions (5.29), we obtain the same critical load (5.33).

It is obvious that if the load q remains normal to the deformed axis of the strip, i.e. it behaves as a normal pressure, the last term in the second equation of (5.37) disappears, and such a loss of stability becomes impossible. In exactly the same way, the last terms in the last equation of (5.26) and Eq. (5.27) for a strip also vanish. The general solution of the equation $M'' = 0$ will be $M = C_1 + C_2x$, and when there are no moments on the ends of the strip we obtain that $C_1 = C_2 = 0$ and $M \equiv 0$. In this case only a shear form of loss of stability, defined by relations (5.30)–(5.32), becomes possible.

The equations and bifurcations of their solutions, corresponding to the use of the complete non-contradictory kinematic relations in the quadratic approximation. In the relations of Hooke's law (5.18) we replace the relation for τ_{xy} as follows:

$$\tau_{xy} = G_{12}\sin\gamma_{12} \tag{5.38}$$

which leads to the relations of elasticity, which differ from (5.19) by the replacement of the last equality by

$$N_{xy} = B_{12}[(1+\varphi)\upsilon' + (1+u')\gamma] \tag{5.39}$$

In the case considered, when using relation (4.8) instead of expression (4.9), we obtain equilibrium equations of the form

$$\begin{gathered} (N_x + N_{xy}\gamma)' = 0, \quad [N_x\upsilon' + N_{xy}(1+\varphi) + M\varphi']' = 0 \\ M' - N_y\gamma - N_{xy}(1+u') + 2hp_{xy} = 0, \quad (M\upsilon')' - N_y - N_{xy}\upsilon' + 2hp_y = 0 \end{gathered} \tag{5.40}$$

for which the boundary conditions when $x = 0$ and $x = a$ take the form

$$\begin{gathered} N_x + N_{xy}\gamma = 2hp_x \quad \text{for} \quad \delta u \neq 0 \\ N_x\upsilon' + N_{xy}(1+\varphi) + M\varphi' = 2p_{xy} \quad \text{for} \quad \delta\upsilon \neq 0 \\ M' = 0 \quad \text{for} \quad \delta\gamma \neq 0, \quad M\upsilon' = 0 \quad \text{for} \quad \delta\varphi \neq 0 \end{gathered} \tag{5.41}$$

If the initial stress–strain state of the strip is defined by the solution

$$N_x^0 = -2hp, \quad N_y^0 = -2hq, \quad N_{xy}^0 = 2h\tau, \quad M^0 = 0 \tag{5.42}$$

Eqs (5.40) and boundary conditions (5.41), linearized in the neighbourhood of this solution, can be written in the form

$$\begin{aligned} &(N_x + 2h\tau\gamma)' = 0, \quad N_y + 2h\tau v' = 0 \; (N_{xy} - 2hpv' + 2h\tau\varphi)' = 0 \\ &M' - N_{xy} + 2hq\gamma - 2h\tau u' = 0 \end{aligned} \tag{5.43}$$

for $x = 0$ and $x = a$

$$\begin{aligned} &N_x + 2h\tau\gamma = 0 \quad \text{or} \quad \delta u = 0; \quad N_{xy} - 2hpv' + 2h\tau\varphi = 0 \quad \text{or} \quad \delta v = 0 \\ &M' = 0 \quad \text{when} \quad \delta\gamma = 0 \end{aligned} \tag{5.44}$$

where, as previously

$$N_x = B_1 u', \quad N_y = B_2\varphi, \quad N_{xy} = B_{12}(v' + \gamma), \quad M = D_1\gamma' \tag{5.45}$$

We will initially consider a strip under pure shear conditions due to the action of forces per unit length $p_{xy} = \tau$, applied at the ends (Fig. 2). In this case $p = q = 0$, and for the first equation of (5.43) we will take the boundary conditions in the form $N_x + 2h\tau\gamma = 0$ when $x = 0$ and $x = a$. By virtue of this, Eqs (5.43), using relations (5.45), can be represented in the form

$$\begin{aligned} &B_1 u' + 2h\tau\gamma = 0, \quad B_2\varphi + 2h\tau v' = 0, \quad B_{12}(v' + \gamma) + 2h\tau\varphi = C_1 \\ &D_1\gamma'' - B_{12}(v' + \gamma) - 2h\tau u' = 0 \end{aligned} \tag{5.46}$$

From the first two equations of (5.46) we obtain

$$u' = -\frac{S}{B_1}\gamma, \quad \varphi = -\frac{S}{B_2}v'; \quad S = 2h\tau \tag{5.47}$$

Consequently, the last two equations of (5.46) take the form

$$\gamma = \frac{C_1}{B_{12}} - \chi v', \quad D_1\gamma'' - \frac{S^2}{B_2}v' + \frac{S_2}{B_1}\gamma = C_1; \quad \chi = 1 - \frac{S^2}{B_2 B_{12}} \tag{5.48}$$

When using the equation of (5.48) we will write the first integral of the second equation in the form

$$v'' + k^2 v = bC_1 x + C_2 \tag{5.49}$$

Here

$$k^2 = \frac{1-\chi}{D_1}\left(\frac{B_{12}}{\chi} + \frac{B_2 B_{12}}{B_1}\right), \quad b = -\frac{1}{D_1} \tag{5.50}$$

Substituting the general solution of Eq. (5.49) into the expression for γ, we obtain, after differentiation with respect to x,

$$\gamma' = \chi k^2(C_3 \sin kx + C_4 \cos kx) \tag{5.51}$$

We will subject the solution (5.51) to the last condition of (5.44) when $x = 0$. We then obtain the equation

$$\chi k^2 C_4 = 0 \tag{5.52}$$

which can be satisfied in three cases:
(1) $C_4 = 0$;
(2) $\chi = 0$, whence we obtain the positive bifurcation value

$$S_*^{(1)} = \sqrt{B_2 B_{12}} \quad \text{or} \quad \tau_*^{(1)} = \sqrt{E_2 G_{12}} \tag{5.53}$$

and also

$$\gamma = C_1/B_{12} = \text{const}, \quad \gamma' \equiv 0 \tag{5.54}$$

i.e. all the section x = const, on changing into the perturbed state, are rotated by an angle that is constant along the length of the strip, and which is characteristic for a shear form of loss of stability; in this case $k = \infty$, and the displacement υ becomes indeterminate, unlike the cases corresponding to the action of the forces p and q;
(3) $k^2 = 0$, which, when using the first formula of (5.50), leads to another positive bifurcation value

$$S_*^{(2)} = \sqrt{(B_1 + B_2)B_{12}}, \quad \text{or} \quad \tau_*^{(2)} = \sqrt{(E_1 + E_2)G_{12}} \tag{5.55}$$

It can be seen that $\tau_*^{(2)} > \tau_*^{(1)}$.

We now subject solution (5.51) to the condition $M' = 0$ when $x = a$ for the case when $C_4 = 0$. Then

$$\chi k^2 \sin ka C_3 = 0 \tag{5.56}$$

Hence if follows that $c_3 \neq 0$, only when, unlike the cases when $\chi = 0$ and $k^2 = 0$, the following condition is satisfied

$$k^2 = n^2\pi^2/a^2, \quad n = 0, 1, 2, \dots$$

Using the first formula of (5.50), we arrive at a quadratic equation in S^2, whence we find

$$S_*^{(3,4)} = (U \pm \sqrt{U^2 - V})^{1/2}; \quad U = \frac{1}{2}[(B_1 + B_2)B_{12} + B_1 T_E], \quad V = B_1 B_2 B_{12} T_E$$
$$T_E = \frac{\pi^2}{a^2}D_1 = \frac{\pi^2 h^2}{a^2}\frac{B_1}{3}n^2 \tag{5.57}$$

Hence it follows that

$$S_*^{(3)} \equiv 0, \quad S_*^{(4)} = S_*^{(2)} \quad \text{when} \quad n = 0; \quad S_*^{(3)} = 0, \quad S_*^{(4)} = S_*^{(1)} \quad \text{when} \quad B_1 \equiv 0$$

An analysis of the roots of (5.57) showed that when $n \neq 0$, $S_*^{(4)}$ is a minimum when $n = 1$.

For the purpose of comparing the quantities $S_*^{(1)}$ and $S_*^{(4)}$, defined by formulae (5.53) and (5.57) (when choosing the minus sign), we set up the relation

$$r^2 = (S_*^{(4)}/S_*^{(1)})^2 = \zeta - \sqrt{\zeta^2 - \eta}$$

where

$$\zeta = \frac{1}{2}(1 + k_1 + \eta), \quad \eta = \frac{\pi^2}{3}k_1 k_{12}\varepsilon^2, \quad k_1 = \frac{B_1}{B_2}, \quad k_{12} = \frac{B_1}{B_{12}}, \quad \varepsilon = \frac{h}{a}$$

The table shows values of r for different values of k_1, k_{12} and ε; for $\varepsilon = 0.5$, generally speaking, not corresponding to the case when $\varepsilon \ll 1$, the values of r lie between 0.938 and 0.999. It can be seen that $S_*^{(4)} < S_*^{(1)}$ always. For small values of the stiffness on the transverse shear ($B_{12}/B_1 \ll 1$) for a strip of average thickness ($\varepsilon = 0.1$) the values of $S_*^{(1)}$ approach the values of $S_*^{(4)}$ when the parameter $k_1 = B_1/B_2$ increases.

Table 1

k_1	$k_{12} = 10$			10^2			10^3		
	ε								
	0.5	0.1	0.01	0.5	0.1	0.01	0.5	0.1	0.01
1	0.938	0.388	0.040	0.993	0.848	0.127	0.999	0.984	0.388
10	0.943	0.483	0.054	0.994	0.873	0.171	0.999	0.985	0.484

Hence, for pure shear, solution (5.51) corresponds to the least bifurcation value τ_*; it is necessary to take $C_4 = 0$ in this solution. In this case we have the following function for v

$$v = C_3 \sin kx + \frac{1}{k^2}(bC_1 x + C_2) \tag{5.58}$$

and for γ and γ' we have the functions

$$\gamma = \left(\frac{1}{B_{12}} - \frac{b}{k^2}\chi\right)C_1 - C_3 \chi k \cos kx; \quad \gamma' = \chi k^2 C_3 \sin kx \tag{5.59}$$

by which the corresponding form of the loss of stability is described, after determining two of the three constants of integration.

Note that, for a rectangular orthotropic plate, subjected to pure shear, the solution of the problem of the shear form of loss of stability was given previously in [5], starting from the linearized equations of the plane problem of the theory of elasticity, the basis of which is the use of the complete kinematic relation (4.3). The bifurcation value τ^* for an isotropic plate obtained in it is equal to

$$\tau_* = G = \frac{E}{2(1+\nu)}$$

which, for $\nu = 0.3$ is 1.6 times less than the value of τ_*, found from formula (5.53).

It is easy to show that when a force p acts on the strip, when $q = \tau = 0$, following from problem (5.43)–(5.45), the problem is identical with that considered above, based on the most simplified equations, and when a force q acts, when $p = \tau = 0$, the problem, which follows from problem (5.43)–(5.45), is identical with that investigated above based on the partially simplified equations.

6. ANALYSIS OF THE KINEMATIC RELATIONS IN ORTHOGONAL CURVILINEAR COORDINATES

We refer to continuum of the underformed body to an orthogonal system of curvilinear coordinates x^α ($\alpha = 1, 2, 3$), in which the Lamé parameters H_α and the unit vectors $\mathbf{l}_\alpha$ are defined. If the displacement vector U of an arbitrary point $M(x^\alpha)$ is represented by the expansion $U = U_\alpha \mathbf{l}_\alpha$, then in the deformed state for the unit vectors $\mathbf{l}^*_\alpha$, and also for the elongation deformation E_α and the shear deformation $\gamma_{\alpha\beta}$ ($\beta = 1, 2, 3$) without introducing any limitations on their values, there are analogies with the formulae derived in Section 1 (summation, in accordance with the generally accepted rules, is carried out over dummy indices)

$$\begin{aligned} \mathbf{l}^*_\alpha &= (\delta_{\alpha\beta} + e_{\alpha\beta})\mathbf{l}_\beta / h^*_\alpha, \quad E_\alpha = h^*_\alpha - 1 \\ \sin\gamma_{\alpha\beta} &= 2\varepsilon_{\alpha\beta}/(h^*_\alpha h^*_\beta) \ (\alpha \neq \beta); \quad h^*_\alpha = (1 + 2\varepsilon_{\alpha\alpha})^{1/2} \end{aligned} \tag{6.1}$$

Here

$$\begin{aligned} 2\varepsilon_{\alpha\beta} &= e_{\alpha\beta} + e_{\beta\alpha} + e_{\alpha\delta}e_{\beta\delta} = \delta_{\beta\pi}e_{\alpha\pi} + \delta_{\alpha\pi}e_{\beta\pi} + e_{\alpha\pi}e_{\beta\pi} = \\ &= \left(\delta_{\alpha\pi} + \frac{e_{\alpha\pi}}{2}\right)e_{\beta\pi} + \left(\delta_{\beta\pi} + \frac{e_{\beta\pi}}{2}\right)e_{\alpha\pi} \end{aligned} \tag{6.2}$$

$$e_{11} = \frac{1}{H_1}\frac{\partial u_1}{\partial x^1} + \frac{1}{H_1 H_2}\frac{\partial H_1}{\partial x^2}u_2 + \frac{1}{H_1 H_3}\frac{\partial H_1}{\partial x^3}u_3$$
$$e_{12} = \frac{1}{H_1}\frac{\partial u_2}{\partial x^1} - \frac{1}{H_1 H_2}\frac{\partial H_1}{\partial x^2}u_1, \quad e_{13} = \frac{1}{H_1}\frac{\partial u_3}{\partial x^1} - \frac{1}{H_1 H_3}\frac{\partial H_1}{\partial x^3}u_1, \quad \overrightarrow{\underleftarrow{1, 2, 3}} \tag{6.3}$$

The analogues of formulae (1.3)

$$E_\alpha \approx \varepsilon_{\alpha\alpha} \tag{6.4}$$

simplified for the case of small elongation deformation, are generally accepted; following from the first equality of (6.1) with an accuracy of $2 + E_\alpha \approx 2$, and from the second equality of (6.1) with an accuracy of $(1 + 2\varepsilon_{\alpha\alpha})^{-1/2} \approx 1$ we obtain the approximate formulae

$$\gamma_{\alpha\beta} \approx 2\varepsilon_{\alpha\beta} \ (\alpha \neq \beta) \tag{6.5}$$

if we take $\sin\gamma_{\alpha\beta} \neq \gamma_{\alpha\beta}$

We will assume that, at each point of the deformed body, the x^α axes are the principal axes of deformation. In this system of coordinates

$$\gamma_{\alpha\beta} = 2\varepsilon_{\alpha\beta} = 0 \ (\alpha \neq \beta)$$

and only in the case when $e_{\alpha\beta} = 0$ when $\varepsilon \neq \beta$. But in this case, as follows from Eq. (6.2), the following expressions hold

$$2\varepsilon_{\alpha\alpha} = 2e_{\alpha\alpha} + e_{\alpha\alpha}^2$$

Substitution of these expressions into the second equality of (6.1) leads to the exact formulae

$$E_\alpha = (1 + 2e_{\alpha\alpha} + e_{\alpha\alpha}^2)^{1/2} - 1 = e_{\alpha\alpha} \tag{6.6}$$

which are analogues of formulae (2.3), whereas within the framework of the approximate formulae (6.4) we arrive at the result

$$E_\alpha \approx \varepsilon_{\alpha\alpha} = e_{\alpha\alpha} + e_{\alpha\alpha}^2/2 \tag{6.7}$$

In addition to the problems considered previously in [4, 5], we will show below, using the simplest example, to what physically incorrect results and conclusions one can be led by using approximations (6.4) when solving specific problems.

7. SOLUTIONS OF THE PROBLEM OF THE NEUTRAL EQUILIBRIUM OF A CIRCULAR RING UNDER A UNIFORM EXTERNAL PRESSURE HAVING ZERO VARIABILITY IN A CIRCUMFERENTIAL DIRECTION

We will consider the solutions of the problem of the neutral equilibrium (forms of loss of stability) of a circular ring when acted upon by a uniform external pressure, having zero variability of the parameters of the perturbed stress–strain state in a circumferential direction. The forms of loss stability and the vibrations of three-layered structures, which have zero variability in a circumferential direction have been described and investigated previously in [4, 10, 11].

The unperturbed (subcritical) stress–strain state. Suppose a circular ring of orthotropic material, having a thickness $2h$ and a radius of the middle surface R, is in a plane stress–strain state under the action of a uniform external pressure p. If the middle surface of the ring is referred to a circumferential (angular) coordinate θ and a radial coordinate $z(-h \leq z \leq h)$, connected with the dimensionless coordinate π by the relation $\rho = 1 + z/R$, its unperturbed (axisymmetrical) stress–strain state will be described by the equilibrium equation

$$\frac{d\sigma^0_{zz}}{d\rho}+\frac{\sigma^0_{zz}-\sigma^0_{\theta\theta}}{\rho} = 0 \tag{7.1}$$

Here and henceforth all the parameters of the unperturbed stress–strain state are denoted by a zero superscript.

If the material of the ring is linearly elastic, with elastic parameters $\tilde{E}_2$, $\tilde{E}_3$, G_{23}, ν_2, ν_3, where $\tilde{E}_2\nu_2 = \tilde{E}_3\nu_3$, the circumferential and radial normal stresses in Eq. (7.1) are connected with the corresponding radial displacement u_2 by the relations of the generalized Hooke's law

$$\sigma^0_{zz} = \frac{E^*_3}{R}\left(\frac{du^0_z}{d\rho}+\nu_3\frac{u^0_z}{\rho}\right),\ \sigma^0_{\theta\theta} = \frac{\tilde{E}_2}{R}\left(\frac{u^0_z}{\rho}+\nu_2\frac{du^0_z}{d\rho}\right);\ E^*_3 = \frac{\tilde{E}_3}{(1-\nu_2\nu_3)},\ E^*_2 = \frac{\tilde{E}_2}{(1-\nu_2\nu_3)} \tag{7.2}$$

It should be noted that in the case considered the θ and z axes are the principal axes of deformation and the following equalities are strictly satisfied

$$E^0_z = E^0_{zz},\quad E^0_\theta = E^0_{\theta\theta}$$

in which

$$E^0_{zz} = du^0_z/dz = R^{-1}du^0_z/d\rho,\quad E^0_{\theta\theta} = R^{-1}u^0_z/\rho$$

Hence, the elasticity relations (7.2) and equilibrium equation (7.1), in which the stresses are referred to the corresponding areas in the undeformed state of the ring, are also exact.

After substituting expressions (7.2) into the equilibrium equation (7.1) we arrive at an equation in u_z.

$$\frac{d^2u^0_z}{d\rho^2}+\frac{du^0_z}{\rho\partial\rho}-\delta^2\frac{u^0_z}{\rho^2} = 0;\quad \delta^2 = \frac{\tilde{E}_2}{\tilde{E}_3}$$

the solution of which with the boundary conditions

$$\sigma^0_{zz} = -p \quad\text{when}\quad z = h,\quad \sigma^0_{zz} = 0 \quad\text{when}\quad z = -h$$

has the form

$$\begin{gathered} u^0_z = A_+\rho^\delta + A_-\rho^{-\delta},\quad A_\pm = \frac{pR}{\Delta E^*_3(\delta\pm\nu_3)}\rho_1^{\mp\delta-1} \\ \Delta = \rho_1^{\delta-1}\rho_2^{-\delta-1}-\rho_1^{-\delta-1}\rho_2^{\delta-1},\quad \rho_1 = 1-h_0,\quad \rho_2 = 1+h_0,\quad h_0 = h/R \end{gathered} \tag{7.3}$$

By formulae (7.2) ad (7.3) we have the following expressions for the stresses

$$\sigma^0_{zz} = \frac{p}{\rho_1\rho\Delta}\left[\left(\frac{\rho}{\rho_1}\right)^\delta-\left(\frac{\rho_1}{\rho}\right)^\delta\right],\quad \sigma^0_{\theta\theta} = \frac{p\delta^2}{\rho_1\rho\Delta}\left[\frac{1+\nu_2\delta}{\delta+\nu_3}\left(\frac{\rho}{\rho_1}\right)^\delta+\frac{1-\nu_2\delta}{\delta-\nu_3}\left(\frac{\rho_1}{\rho}\right)^\delta\right] \tag{7.4}$$

which, for a ring of isotropic material, by virtue of the equalities

$$\nu_2 = \nu_3 = \nu,\quad \tilde{E}_2 = \tilde{E}_3 = E,\quad \delta = 1$$

take the form

$$\frac{\sigma^0_{zz}}{\rho^2-\rho_1^2} = \frac{\sigma^0_{\theta\theta}}{\rho^2+\rho_1^2} = -\frac{p\rho_2^2}{4h_0\rho^2} \tag{7.5}$$

In the other hypothetically possible case, when $\delta \ll 1$ ($\tilde{E}_2 \ll \tilde{E}_3$), as $\delta \to 0$ we obtain from relations (7.4)

$$\frac{\sigma_{zz}^0}{\ln(\rho/\rho_1)} = \sigma_{\theta\theta}^0 = -\frac{p\rho_2/\rho}{\ln(\rho_2/\rho_1)} \tag{7.6}$$

For a thing ring, when $h_0 \ll 1$, from both relations (7.5) and (7.6) we obtain the same approximate formulae

$$\frac{\sigma_{zz}^0}{(z_0 + h_0)} = \sigma_{\theta\theta}^0 = -\frac{p}{2h_0}; \quad z_0 = \frac{z}{R} \tag{7.7}$$

from which we see that $\sigma_{\theta\theta}^0 = \sigma_{zz}^0/h_0$ when $z_0 = h_0$.

Forms of neutral equilibrium of a ring with zero variability of the parameters of the perturbed stress–strain state. If the deformation of the ring in the unperturbed state is assumed to be small, we can make the standard assumption that, in this state, it is stressed but not deformed, and to describe its neutral equilibrium with parameters of the perturbed stress–strain state, having zero variability in the circumferential direction (i.e. $\partial/\partial\theta = 0$), using the results obtained by Novozhilov, Guz' and others (see, for example, [1]), we can set up the following system of homogeneous differential equations

$$\frac{d}{\rho d\rho}(\rho\tilde{\sigma}_{z\theta}) + \frac{\tilde{\sigma}_{\theta z}}{\rho} = 0, \quad \frac{d}{\rho d\rho}(\rho\tilde{\sigma}_{zz}) - \frac{\tilde{\sigma}_{\theta\theta}}{\rho} = 0 \tag{7.8}$$

in which $\tilde{\sigma}_{zz}$, $\tilde{\sigma}_{z\theta}$, $\tilde{\sigma}_{\theta z}$, $\tilde{\sigma}_{\theta\theta}$ are the perturbed components of the stresses in the $\mathbf{l}_2$ and $\mathbf{l}_3$ axes of the undeformed state of the ring, connected with the perturbed components of the stresses σ_{zz}, $\sigma_{\theta z} = \sigma_{z\theta}$, $\sigma_{\theta\theta}$ in the deformed axis $\mathbf{l}_2^*$ and $\mathbf{l}_3^*$ and stresses $\sigma_{\theta\theta}^0$, σ_{zz}^0 by the relations

$$\tilde{\sigma}_{zz} = \sigma_{zz} + \sigma_{zz}^0 E_{zz}, \quad \tilde{\sigma}_{\theta\theta} = \sigma_{\theta\theta} + \sigma_{\theta\theta}^0 E_{\theta\theta} \tag{7.9}$$

$$\tilde{\sigma}_{z\theta} = \sigma_{z\theta} + \sigma_{zz}^0(E_{z\theta}/2 - \omega), \quad \tilde{\sigma}_{\theta z} = \sigma_{\theta z} + \sigma_{\theta\theta}^0(E_{z\theta}/2 + \omega) \tag{7.10}$$

where

$$\sigma_{zz} = E_3^*(E_{zz} + \nu_3 E_{\theta\theta}), \quad \sigma_{\theta\theta} = E_2^*(E_{\theta\theta} + \nu_2 E_{zz}), \quad \sigma_{\theta z} = G_{23}E_{\theta z} \tag{7.11}$$

$$E_{z\theta} = \frac{du_\theta}{Rd\rho}, \quad E_{zz} = \frac{du_z}{Rd\rho}, \quad E_{\theta\theta} = \frac{u_\theta}{R\rho}, \quad E_{\theta z} = \frac{1}{R}\left(\frac{du_\theta}{d\rho} - \frac{u_\theta}{\rho}\right), \quad \omega = -\frac{1}{2R}\left(\frac{du_\theta}{d\rho} + \frac{u_\theta}{\rho}\right) \tag{7.12}$$

By relations (7.9)–(7.12), Eqs (7.8) reduces to two unconnected ordinary differential equations

$$(G_{23} + \sigma_{zz}^0)\frac{d^2u_\theta}{d\rho^2} + \frac{G_{23} + \sigma_{\theta\theta}^0}{\rho}\left(\frac{du_\theta}{d\rho} - \frac{u_\theta}{\rho}\right) = 0 \tag{7.13}$$

$$(E_3^* + \sigma_{zz}^0)\frac{d^2u_z}{d\rho^2} + \frac{E_3^* + \sigma_{\theta\theta}^0}{\rho}\frac{du_z}{d\rho} - \frac{E_2^* + \sigma_{\theta\theta}^0}{\rho^2}u_z = 0 \tag{7.14}$$

Equations (7.13) and (7.14) for $z = \pm h$ must respectively satisfy the following conditions

$$\tilde{\sigma}_{z\theta}(\rho = \rho_1) = 0, \quad \tilde{\sigma}_{z\theta}(\rho = \rho_2) = 0 \tag{7.15}$$

$$\tilde{\sigma}_{zz}(\rho = \rho_1) = 0, \quad \tilde{\sigma}_{zz}(\rho = \rho_2) = 0 \tag{7.16}$$

The equations compiled, with boundary conditions (7.15) and (7.16), can be investigated with different degrees of approximation. However, the principal conclusions can also be formulated when they are investigated for a thin ring using the simplest shell model

$$u_z \approx w, \quad u_\theta \approx v + \rho\psi \tag{7.17}$$

which has been called a Timoshenko-type model, ignoring transverse compression. Here w and v are the radial and circumferential displacements of points on the axis of the ring.

Application of the first relation of (7.17) to Eq. (7.14) leads to the equality

$$w\int_{\rho_1}^{\rho_2}(E_2^* + \overset{0}{\sigma}_{\theta\theta})d\rho = 0$$

which, using the second formula of (7.7) for a thin ring, gives $w(E_2^* - p) = 0$. Hence follows the bifurcation value of the external pressure $p^* = E_2^*$, similar to that obtained previously in [4, 5] and related to the assumption of the approximation $E_\theta \approx \varepsilon_{\theta\theta}$ in the initial geometrically non-linear problem, formulated using general considerations of the form

$$E_\alpha \approx \varepsilon_{\alpha\alpha}, \quad \gamma_{\alpha\beta} \approx 2\varepsilon_{\alpha\beta} \ (\alpha \neq \beta)$$

If, instead of $u_z \approx w$ we use the more accurate approximations and satisfy conditions (7.16), we can establish from Eq. (7.14) several bifurcation values of the external pressure by using the approximations

$$E_\theta \approx \varepsilon_{\theta\theta}, \quad E_z \approx \varepsilon_{zz}$$

when constructing the initial geometrically non-linear equations.

Applying the second relation of (7.17) to Eq. (7.13) and the second formula of (7.7), we obtain

$$\int_{\rho_1}^{\rho_2}\frac{(G_{23} + \overset{0}{\sigma}_{\theta\theta})}{\rho}v d\rho \approx 2\left(G_{23} - \frac{p}{2h_0}\right)vh_0 = 0$$

Hence, by virtue of the fact that $v \neq 0$, we obtain the bifurcation value of the external pressure

$$p_* = 2h_0G_{23} \tag{7.18}$$

which, in the light of the results obtained previously in [4, 10], is the critical value of the external pressure, on reaching which the ring loses stability with respect to a purely shear form. This form of loss of stability of the ring occurs physically when $\overset{0}{\sigma}_{\theta\theta}$ reaches the values G_{23}, if a purely flexural form of loss of stability does not occur earlier for lower values of p.

8. NON-CONTRADICTORY KINEMATIC RELATIONS IN THE QUADRATIC APPROXIMATION FOR THE CASE OF SMALL ELONGATION DEFORMATION AND AVERAGE SHEAR DEFORMATION

In the light of the results described for the case of small elongation deformation ($E_\alpha \approx \varepsilon$) in orthogonal curvilinear coordinates the following relations are more correct compared with the relations used in the literature

$$E_1 \approx e_{11} + (e_{12}^2 + e_{13}^2)/2, \ldots \tag{8.1}$$

which are an analogue of relations (1.6) and, unlike (6.7), enables one, in the limit, to transfer to formulae (6.6). Then, to determine the shear deformation it is necessary to use the relations

$$\sin\gamma_{12} \approx 2\varepsilon_{12} = e_{12}(1 + e_{22}) + e_{21}(1 + e_{11}) + e_{13}e_{23}, \ldots \tag{8.2}$$

which are analogues of relations (1.4).

It should be noted that relations (8.1) follow from (6.2) and (6.4) when and only when, in addition to satisfying the estimates $E_\alpha \approx \varepsilon$ (i.e. when one can use approximation (6.4)), the following estimates are satisfied

$$e_{\alpha\alpha} \approx \varepsilon, \quad e_{\alpha\beta} \approx \sqrt{\varepsilon}\ (\alpha \neq \beta) \tag{8.3}$$

In other words, the elongation deformation can only be small when the quantities $e_{\alpha\alpha}$ are small and are "average" (i.e. of the order of $\sqrt{\varepsilon}$) quantities $e_{\alpha\beta}$ ($\alpha \neq \beta$). When $h_\alpha^* \approx 1$ the geometrical meaning of the latter can easily be established from the first equality of (6.1), for example $e_{12} = \cos(\mathbf{l}_1^*, \mathbf{l}_2^*)$, etc. Since the shear deformation is defined in terms of the quantities $e_{\alpha\beta}$ ($\alpha \neq \beta$), and, when the estimates (8.3) are satisfied in accordance with (8.2), they are "average", relations (8.2), within the framework of the approximations (5.1) employed, allow of further simplifications of the form

$$\gamma_{12} \approx \sin\gamma_{12} \approx 2\varepsilon_{12} \approx e_{12} + e_{21} + e_{13}e_{23}, \ \ldots \tag{8.4}$$

Finally, for small elongation deformation and average shear deformation the kinematic relations in the quadratic approximation (8.1) and (8.4) are correct and are justified with the necessary degree of completeness. The geometrically non-linear equations of the theory of elasticity, formulated using them, enable us to investigate only physically realizable forms of loss of stability which, in particular, where demonstrated in [4].

For the case of the deformed state considered, expressions (6.1) take the form

$$\mathbf{l}_1^* = \mathbf{l}_1 + e_{12}\mathbf{l}_2 + e_{13}\mathbf{l}_3, \ \overrightarrow{\underleftarrow{1, 2, 3}}$$

Using them and starting from the representations

$$\boldsymbol{\sigma}_\alpha = \sigma_{\alpha\beta}\mathbf{l}_\beta^* = \tilde{\sigma}_{\alpha\beta}\mathbf{l}_\beta$$

one can establish the following relations between the components of the stresses $\sigma_{\alpha\beta}$ and $\tilde{\sigma}_{\alpha\beta}$

$$\begin{aligned} \tilde{\sigma}_{11} &= \sigma_{11} + \sigma_{12}e_{21} + \sigma_{13}e_{31}, \quad \tilde{\sigma}_{12} = \sigma_{11}e_{12} + \sigma_{12} + \sigma_{13}e_{32} \\ \tilde{\sigma}_{13} &= \sigma_{11}e_{13} + \sigma_{12}e_{23} + \sigma_{13}, \quad \overrightarrow{\underleftarrow{1, 2, 3}} \end{aligned} \tag{8.5}$$

which are correct and do not lead to "absurd" force boundary conditions. We can also show this by considering the problem investigated in Section 7, from the positions described. In fact, when using the kinematic relations (8.1) and (8.4) as well as relations (8.5) to formulate the initial geometrically non-linear equations, we must introduce the following formulae into the equations of neutral equilibrium (7.8) instead of (7.9)

$$\tilde{\sigma}_{zz} = \sigma_{zz}, \quad \tilde{\sigma}_{\theta\theta} = \sigma_{\theta\theta} \tag{8.6}$$

in which σ_{zz} and $\sigma_{\theta\theta}$ are defined, as before, from formulae (7.11). As a result the "strange" boundary conditions (7.16) become clear:

$$\sigma_{zz}(\rho = \rho_1) = 0, \quad \sigma_{zz}(\rho = \rho_2) = 0 \tag{8.7}$$

and, instead of Eq. (7.14), we arrive at the equation

$$E_3^*\left(\frac{d^2u_z}{d\rho^2} + \frac{du_z}{\rho d\rho}\right) - \frac{E_2^*}{\rho^2}u_z = 0$$

which, in view of boundary conditions (8.7), only have a trivial solution $u_z \equiv 0$.

9. CONCLUSIONS

1. If a uniaxial stress of longitudinal compression is formed in the components of structures, the form of loss of stability which occurs in them is either purely flexural or purely shear. To determine the

bifurcation values of the acting load in this case for elongation deformation in the direction of the compression it is necessary to use a kinematic relation in the form (1.6). However, for deformation in directions orthogonal to the compression, and for shear, it is permissible to use linear kinematic relations.

2. For a uniform transverse compression, the behaviour of the load turns out to be important. If it remains normal to the deformed axis, bifurcation is only possible in shear form. If the load maintains its direction, then, in addition to a shear form of bifurcation a flexural form of loss of stability is possible, identical in form with the Euler form, for which there is no shear.

3. If a stress state, close to pure shear, is formed in the structural component, then to investigate the bifurcation values of the load, shear deformation is necessary to describe the non-linear kinematic relations in the complete quadratic version of the form (1.5).

4. For thin-walled structural components in the form of rods, for thin-walled structural components in the form of rods, plates and shells, made of composite materials, acceptable results in determining the form of loss of stability and the critical loads can be expected from using the simplest improved models, known in the literature and constructed taking into account the transverse shear. At the same time, in the light of the results obtained, further investigation of the problems related to improving the non-linear theory of multi-layer structural components (in particular, three-layer components), the structure of which is formed from alternating thin and rigid layers and layers of filler of lower rigidity, is primarily required. For three-layer shells with fillers, which belong to the class of transversely weak structures [12], these non-contradictory versions of the theory, which are interesting in view of the possibility of investigating shear forms of loss of stability when compression stresses are formed in the filler in a transverse direction, were constructed previously in [4].

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